2 As phony as a $3 bill? Coin and note denominations

If you are a central bank, how do you design an optimal set of coins and notes? In particular, what denominations should you select? As we saw in the previous chapter, cash payments span multiple orders of magnitude: from 0.01 cent to hundreds of dollars. The denominations should allow for paying all such amounts in a convenient way.

This problem has, of course, been solved numerous times in practice. Table 1 shows the Roman coin system that was largely based on powers of 2.

Table 1: Roman coins denominations, Augustan values

Name	Value in Sertertius	Value in Quadrans
Aureus	100	1600
Quinarius Aureus	50	800
Cistophorus	12	192
Antonianus	8	128
Denarius	4	64
Quinarius Argenteus	2	32
Sertertius	1	16
Dupondis	½	8
As	¼	4
Semis	1/8	2
Quadrans	1/16	1

At the time of Charlemagne, the penny was the dominant minted coin and contained about 1.7g of silver. Large quantities of pennies were counted in dozens (the shilling) and score dozens (the pound).[1] A British pound sterling being 240 pennies thus was indeed a pound of silver. This system was still used by the UK, prior to going decimal in 1971. It has been argued that this system was close to powers of 3 (Table 2).[2]

Table 2: Denominations of UK coins and notes (in pennies)

[1] Sargent and Velde (1997), p17.

[2] Telser (1995) converts all denomination to pence, so that 1 shilling = 12 pence; ½ crown = 2½ shilling = 30 pence, etc. He omits the farthing (¼ pence), half-penny and two-pence.

Powers of 3:	1	3	9		27	81	243
UK prior to '71:	1	3	6	12	30	60	240

Most of today's systems are decimal in the sense that they have denominations for the powers of 10 ($^1/_{100}$, $^1/_{10}$, 1, 10 etc.). To fill the gap between these powers of ten, most currencies use some form of the so called *binary-decimal triplet* {1, 2, 5}. Some 20 currencies use this system in its pure form, including the euro which has 5 of these triplets going from a 1 Eurocent coin all the way to a 500 euro note. Less common, but still frequently used, are the *fractional-decimal triplet* {1, 2½, 5} and decimal pairs like {1, 5}, {1, 2½}. Many currencies use a mixture of these. The US dollar, for example, has a 25¢ coin, a $2 and $20 note but neither a 50¢ coin nor a $50 bill. And there is, of course, no $3 bill. In fact, only very few currency systems have coins or bills that are powers or multiples of 3.[3]

So we know it works in practice, but does it work in theory? Are these systems indeed optimal? This problem turns out to be harder than it looks. It has inspired significant modelling effort and some fierce academic debate between two different schools of thought.

The first school looks for the minimal set of different denominations that can make any payment, assuming each denomination (like weights) can be used only once in a payment. It turns out the optimal denominations are the powers of 3, hence the old UK system came close.

A second school of thought looks at the *total* number of tokens (coins and notes) needed for a transaction, i.e. allowing for the use of multiple coins of the same denomination. If exact payment is required then it can be shown that the system with base 2 is the most efficient, i.e. yields the lowest number of

[3] A 3 lek note in Albania, a 3 peso note in Cuba, a 3 bani coin in Rumania, a 3 rouble note in Russia and 3 Bahamian dollar note. These are the exceptions, not the rule. Wynne (1997).

tokens across all transaction sizes.[4] Under this view, the Romans got it right.

Blâchet's weight problem

Finding the minimal set of different denominations that can make any payment (allowing for change) is mathematically related to the weight problem of Bâchet: break a 40 kg stone into as few pieces as possible so that you can weigh any whole-kg amount between 1 and 40 kg using only a two-scale balance.[5] The answer is to use powers of 3 and break the stone into 4 pieces weighing 1, 3, 9 and 27 kg.

The analogy to payments is as follows: the object to be weighted corresponds to a transaction price which has to be paid in cash. The weights correspond to the coins and notes used for payment, and the weights added to opposite pan (the pan holding the object to be weighted) correspond to change given in the transaction.

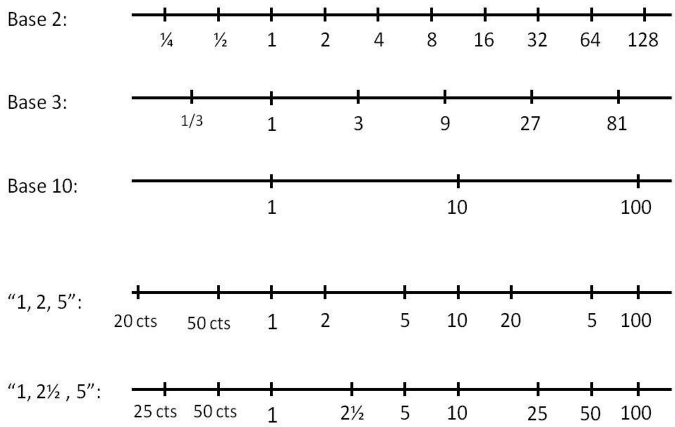
By this analogy, the best denominations for coins and notes would be powers of three: 1, 3, 9, 27, 81, etc.

Things are more complicated if change can be given. Quite some effort has been put in computer simulations to find the system that would result in the lowest number of tokens (coins or notes) across a range of transaction sizes. It turns out that the Roman system (powers of 2) is more efficient than the old English system (powers of 3).[6] But the best system would be based on powers of 1.53, which gives something like {1, 2, 3, 5, 8, 12, 19, 30, 46, etc.}.[7] If this seems arithmetically challenging, there is hope: there is another theoretical optimum for a system that uses powers of 2.16. Using this value yields denominations that are very close to our decimal systems (see inset).

[4] Caianiello, Scarpetta et al. (1982).
[5] Telser (1995).
[6] Van Hove and Hendels (1996).
[7] Bouni and Houy (2007). Denominations are rounded down to the next integer.

Even spacing on logarithmic scales

As was argued in the introduction, logarithms offer a convenient way to deal with variables, such as payment size, that span multiple orders of magnitude. Note how most currency systems are evenly spaced on logarithmic scales:



A systems that would evenly space 2 extra denominations between the powers of 10 would use powers of $\sqrt[3]{10} \approx 2.154$; both the {1, 2, 5} and the {1, 2½, 5} systems get close to this.

There is of course something to be said for ease of arithmetic, which may have played a big role in favour of systems that include multiples of 10. It seems amazing that both the Romans and the English ruled the world with denominations that most of us would find arithmetically challenging.

One thing seems conspicuously absent in most of the research: the actual distribution of payment transaction sizes. The research covered in this chapter assumes transaction sizes are uniformly distributed. In fact, transaction sizes follow a Log-normal distribution which is heavily skewed towards smaller sizes: the modal (most frequent) payment size is close to $2. One could ask, therefore, why the US has no coins for 2 cents and 50 cents, while it does have $20 and $50 notes.